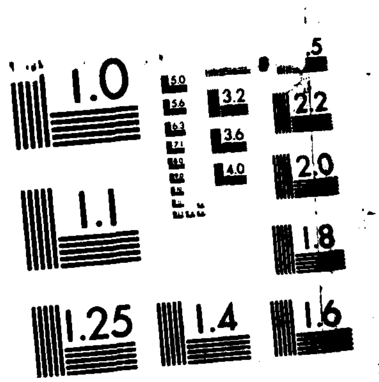
AD-A166 221 HOMOGENEITY AND THE STRONG MARKOV PROPERTY(U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROCESSES O KALLENBERG NOV 85 TR-130 AFOSR-TR-86-0014 1/1

UNCLASSIFIED F49620-85-C-0144 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AFOSR-TR- 86-0014

(2)

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

HOMOGENEITY AND THE STRONG MARKOV PROPERTY

by

Olav Kallenberg

Technical Report No. 130

November 1985

4

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:

3. DISTRIBUTION/AVAILABILITY OF REPORT: Unlimited. Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 130

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 86-0014

6a. NAME OF PERFORMING ORGANIZATION: Center for Stochastic Processes

6b. OFFICE SYMBOL *(If applicable)*:

7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research

6c. ADDRESS *(City, State and ZIP Code)*:
Statistics Dept.,
University of North Carolina
Phillips Hall, 039-A, Chapel Hill, NC 27514

7b. ADDRESS *(City, State and ZIP Code)*:
Bolling Air Force Base
Washington, DC 20332

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR

8b. OFFICE SYMBOL *(If applicable)*: NM

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620-85-C-0144

8c. ADDRESS *(City, State and ZIP Code)*:
Bolling Air Force Base
Washington, DC 20332

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE *(Include Security Classification)*: "Homogeneity and the strong Markov property"

12. PERSONAL AUTHOR(S): Olav Kallenberg

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
technical	FROM 9/85 TO 8/86	Nov. 1985	40

16. SUPPLEMENTARY NOTATION: Stopping times, conditional independence, regeneration, recurrence, invariant σ-fields, exchangeable sequences.

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*:

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

The strong Markov property of a process X at a stopping time τ may be split into a conditional independence part (CI) and a homogeneity part (H). However, it turns out that (H) often implies at least some version of (CI). In the present paper, we shall assume that (H) holds on the set $\{X_\tau \in B\}$, for all stopping times τ such that $X_\tau \to F$ a.s., where F is a closed recurrent subset of the state space S, while $B \subset F$. If F=S, then (CI) will hold on $\{X_\tau \in B\}$ for every stopping time τ, so in this case X is regenerative in B. In the general case, the statement is conditionally true in a suitable sense, given some shift invariant σ-field.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL: Dr Jacobs

22b. TELEPHONE NUMBER *(Include Area Code)*: (202) 767-4940

22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

HOMOGENEITY AND THE STRONG MARKOV PROPERTY

by Olav Kallenberg

Universities of Göteborg, Uppsala, and North Carolina at Chapel Hill

is a member of

Abbreviated title: Homogeneity and the Markov property

Summary:

The strong Markov property of a process X at a stopping time tau τ may be split into a conditional independence part (CI) and a homogeneity part (H). However, it turns out that (H) often implies at least some version of (CI). In the present paper, we shall assume that (H) holds on the set $\{X_\tau \in B\}$, for all stopping times tau τ such that $X_\tau \in F$ a.s., where F is a closed recurrent subset of the state space S, while $B \subset F$ (is a proper subset of). If F=S, then (CI) will hold on $\{X_\tau \in B\}$ for every stopping time tau τ, so in this case X is regenerative in B. In the general case, the same statement is conditionally true in a suitable sense, given some shift invariant delta σ-field.

AMS 1980 subject classifications. 60J25, 60G40.

Key words and phrases. Stopping times, conditional independence, regeneration, recurrence, invariant σ-fields, exchangeable sequences.

Research supported by the Air Force Office of Scientific Research Contract No. F49620 85 C 0144.

1. Introduction. The strong Markov property of a process X at a stopping time τ,

(SM) $\quad P[\theta_\tau \circ X \in \cdot \,|\, \mathcal{F}_\tau] = Q^{X_\tau}$ a.s.,

may be split into a conditional independence part,

(CI) $\quad P[\theta_\tau \circ X \in \cdot \,|\, \mathcal{F}_\tau] = P[\theta_\tau \circ X \in \cdot \,|\, X_\tau]$ a.s.

and a homogeneity part,

(H) $\quad P[\theta_\tau \circ X \in \cdot \,|\, X_\tau] = Q^{X_\tau}$ a.s.

(See Section 2 for notation.) However, it is known that condition (H) alone, for all extended valued stopping times τ, implies the strong Markov property (SM) for all τ. (Cf. Blumenthal and Getoor [2], Proposition 8.2.)

A related result was obtained as Corollary 2.5 of [5], where it was shown, in the discrete time case, that recurrence at every state plus the validity of (H) for every finite stopping time will force X to be a Markov chain (and then automatically strong Markov). This can easily be seen directly (as pointed out to me by H. Föllmer and M. Jacobsen), but it was originally deduced from a general result in exchangeability, via a characterization of recurrent and locally homogeneous sequences as mixtures of Markov chains. (In this paper, mixing will always refer to the associated probability measures.) By local homogeneity is meant that $\theta_\sigma \circ X$ and $\theta_\tau \circ X$ should have the same distribution for every pair of stopping times σ and τ such that X_σ and X_τ are non-random and equal. Note that this is the same as condition (H) for all stopping times τ with fixed X_τ. Extensions of these results to the continuous time case were attempted in Section 4 of [5], and a further discussion was given in [6].

The present work emerged from an attempt to find a unified approach to these results, and to develop a general theory, linking the three conditions (SM), (CI) and (H), both in discrete and

continuous time. As it turns out, an appropriate hypothesis is to assume that (H) should hold a.s. in the set $\{X_\tau \in B\}$, for every stopping time τ such that $X_\tau \in F$ a.s. Here F is a closed subset of the state space S, while B is a Borel subset of F, and it is further assumed that X is <u>recurrent</u> in F, in the sense that

$$\sup\{t \geq 0: X_t \in F\} = \infty \quad \text{a.s.} \tag{1.1}$$

For convenience, the above conditions will be labelled H(F,B) .

Note that recurrence holds automatically when F=S. Thus H(S,B) means simply that (H) should hold a.s. on $\{X_\tau \in B\}$ for every <u>finite</u> stopping time τ. As a special case of Theorem 2 below, it will be seen that X is a strong Markov process, whenever H(S,S) is fulfilled, without any further recurrence conditions on X. This result (mentioned already in [6]) improves the characterizations of Markov processes and chains given in [5]. More generally, it will be seen that H(S,B) for an arbitrary B implies <u>regeneracy</u> in B, in the sense that (SM) will hold a.s. in $\{X_\tau \in B\}$ for every stopping time τ.

The situation becomes more complex when F is a proper subset of S. In that case we can only prove that X regenerates in a certain subset B_r of B, to be referred to as the <u>regular</u> part. The regeneracy may fail in the remaining <u>singular</u> part B_s, but Theorem 3 shows that (SM) remains conditionally true in a suitable sense, given the σ-field induced by the shift invariant sets. Indeed, we shall even prove the stronger statement that X is a mixture of processes regenerating in B_s, and we shall give an explicit expression for the associated transition kernel.

To get a proper understanding of these results and their proofs, it is necessary to look at other ways of characterizing the sets B_r and B_s. The original definition is analytical, in terms of the kernel Q, but Theorem 1 gives alternative descriptions in terms of the sample path behavior of X. When both F and B reduce to a single state s, it turns out that B_s=B=$\{s\}$, so in this case X will

be conditionally regenerative at s. Since the invariant σ-field is independent of s, we may conclude, in the context of recurrent processes in countable state spaces, that X is mixed Markov whenever $H(\{s\},\{s\})$ holds for every state s. This is essentially the characterization of local homogeneity mentioned above.

It may seem unsatisfactory to have different descriptions of the behavior in the regular and singular parts of B. In Theorem 4, it will be shown that (SM) is conditionally true in the entire B, given some suitable invariant σ-field. Unfortunately, the associated transition kernel will typically depend on the choice of conditioning σ-field in this case, and there seems to be no natural choice of the latter. Furthermore, it is no longer clear, in general, whether X can be obtained as a mixture of regenerative processes.

As for the organization of the paper, precise statements of the four main results described above will be given in the next section, with the appropriate framework duly specified. The proofs will then be given in Sections 3-7. Finally, we give in Section 8 a simple example, designed to illustrate the various complications which may arise.

2. Main results. Throughout the paper, we shall consider a fixed random process X, defined on some complete probability space $(\Omega,\mathcal{F},P)$. The paths of X are assumed to lie in the space $D=D(R_+,S)$ of right-continuous functions $w: R_+ \longrightarrow S$ with left-hand limits on $(0,\infty)$. Here S is taken to be a separable metric space, and for the last two theorems we shall even assume that S is complete.

The σ-field in S is taken to be the Borel field $\mathcal{S}$, while D is endowed with the σ-field $\mathcal{D}$ generated by the one-dimensional projections $\pi_t: w \longrightarrow w_t$ from D to S. The process X is assumed to be adapted to some right-continuous filtration $\{\mathcal{F}_t\}\subset\mathcal{F}$, such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

As in Section 1, we shall consider some fixed closed and recurrent set $F\subset S$ and some Borel set $B\subset F$, and we shall assume that condition H(F,B) is fulfilled. Here the shift operators θ_t, $t\geq 0$, on D are defined by

$$(\theta_t w)_s = w_{s+t}, \qquad s,t\geq 0, \quad w\in D,$$

and $Q^{\cdot}$ is assumed to be a normal probability kernel from S to D, where normal means that

$$Q^x\{w\in D: w_0=x\} = 1, \qquad x\in S. \tag{2.1}$$

Note that $\theta_\tau \circ X$ is measurable for every stopping time τ, and that the recurrence relation (1.1) defines a measurable event. Standard facts like these will usually be stated without proofs, if mentioned at all, and we refer to Dellacherie and Meyer [4] for details.

A basic role in the sequel will be played by the D-set

$$A = \{w: w_0\in B\} \cap \bigcap_{t>0}\{w: w_t\in\{w_0\}\cup F^c\},$$

consisting of those paths which start in B and whose excursions from the starting point lie entirely outside F. In terms of A, we may now define the regular and singular parts of B as

$$B_r = \{x\in B:\ Q^x A=0\}, \qquad B_s = \{x\in B:\ Q^x A=1\}.$$

Since $A^c=D\setminus A$ belongs to Souslin $\mathfrak{S}$, we have $A\in\mathfrak{S}^*$, while $B_r, B_s\in\mathcal{J}^*$, where the star indicates universal completion. In particular cases, such as when F=S, we have $A\in\mathfrak{D}$ and hence $B_r, B_s\in\mathcal{J}$, which simplifies some of the arguments below. Note, however, that $A\in\mathfrak{D}$ need not be true in general (cf. Dellacherie [3]).

We proceed to state our first main result, which will play a key role throughout the paper. Here we show that the set $B\setminus(B_r\cup B_s)$ is <u>thin</u>, in the sense that X_τ lies a.s. outside that set for every stopping time τ. The sets where X_τ hits B_r or B_s respectively will further be characterized directly in terms of the sample paths of X, without reference to the kernel Q. Finally we show that B_s is equivalent, from the point of view of hitting at stopping times, to a Borel set B_s', and we describe, in terms of the sample paths of X, the random set

$$M = \{t\geq 0;\ X_t\in B_s'\},$$

which will play such an important role in the subsequent proofs.

For technical reasons, we shall allow the stopping times to take infinite values. For an appropriate interpretation in that case, let us introduce an auxiliary <u>coffin state</u> ∂, and define $X_\infty=\partial$ and $\theta_\infty\circ X=\partial$.

THEOREM 1. <u>For every stopping time</u> τ, <u>we have</u>

$$\{\theta_\tau\circ X\in A\} = \{X_\tau\in B_s\} = \{X_\tau\in B\setminus B_r\} \quad \underline{\text{a.s.}} \tag{2.2}$$

<u>Moreover, there exists a Borel set</u> $B_s'\subset B_s$, <u>such that</u>

$$\{t\geq 0:\ \theta_t\circ X\in A\} = \{t\geq 0:\ X_t\in B_s'\} \quad \underline{\text{a.s.}} \tag{2.3}$$

Here are some consequences of a technical nature which will be useful below:

COROLLARY. The random set M is optional. Moreover, M is a.s. right-closed and either empty or unbounded. Finally, we have for every stopping time τ

$$\{X_\tau \in B_s\} = \{\tau \in M\} \quad \text{a.s.}$$

At this point, one might try to simplify the setting by redefining Q on the set $B \setminus (B_r \cup B_s')$, such as to make $Q^x A=0$ hold for all $x \in B \setminus B_s'$. However, this would lead to new complications, in general, since the altered version of Q may not be measurable with respect to $\mathcal{S}$.

The next result shows that X is regenerative at visits to B_r. If $F=S$, then X is a.s. absorbed when it first hits B_s', and the regenerative property extends to the entire B. In the particular case when even $B=S$, X is then a strong Markov process. This is the result alluded to in [6], and proved for discrete time already in [5].

THEOREM 2. Condition (SM) holds a.s. in $\{X_\tau \in B_r\}$ for every stopping time τ. If $F=S$, the validity extends to $\{X_\tau \in B\}$.

The second statement will be slightly improved below. However, it is generally impossible to extend the regeneracy of X to the whole of B, as will be seen from Example 1 in Section 8.

As noted above, X becomes Markov with transition kernel Q, when $B=F=S$. Though the original Q need not satisfy the Chapman-Kolmogorov identity, as required in the standard axiomatic setup, it can be modified to do so under broad conditions, as shown by Walsh [7].

For the remaining results, the state space S will be assumed to be complete, and hence Polish. Then so is D in the Skorohod-Stone topology, which ensures the existence of regular conditional distributions for X.

Before stating the next main result, we shall need to introduce some further notation. Let us first define the hitting time for B_s'

$$z(w) = \inf\{t \geq 0;\ w_t \in B_s'\}, \quad w \in D,$$

and put $\zeta = z \circ X$. Then ζ is a stopping time, so X_ζ is a random variable, and we may define the measure

$$\mu = P\{X_\zeta \in \cdot,\ \zeta < \infty\} \tag{2.4}$$

and the associated Q-mixture

$$Q^\mu = \int Q^x \mu(dx) = E[Q^{X_\zeta};\ \zeta < \infty],$$

where $E[\cdot\,;C]$ denotes P-integration over the set C. Introducing the shift invariant σ-field $\mathcal{J}$ in D, given by

$$\mathcal{J} = \{I \in \mathcal{D}^*;\ \theta_t^{-1} I = I,\ t \geq 0\},$$

we may next define the conditional distribution $Q^\mu[\cdot|\mathcal{J}]$ as the Q^μ-a.e. unique probability kernel from $(D,\mathcal{J})$ to $(D,\mathcal{D})$, satisfying

$$\int_I Q^\mu[\cdot|\mathcal{J}]\,dQ^\mu = Q^\mu(\cdot \cap I), \quad I \in \mathcal{J}.$$

Apart from the shift operators θ_t, we shall also need the <u>killing operators</u> k_t, given for $0 \leq t \leq \infty$ by

$$(k_t w)_s = \begin{cases} w_s, & s < t, \\ \partial, & s \geq t. \end{cases}$$

Note that the mappings k_t and π_t make sense even when t is a function of w. In particular,

$$\pi_z w = w \circ z(w), \quad w \in D.$$

By $\sigma(\pi_z)$ and $\sigma(k_z)$ we shall mean the σ-fields in D generated by the mappings π_z and k_z respectively. A relation between σ-fields of the form $\mathcal{G} \subset \mathcal{G}'$ a.e. m means that $\mathcal{G}$ lies in the m-completion of $\mathcal{G}'$, and similarly, $\mathcal{G} = \mathcal{G}'$ a.e. m means that $\mathcal{G}$ and $\mathcal{G}'$ have the same m-completion.

The process X is said to be <u>conditionally regenerative</u> in a certain set, if X is regenerative in the set under almost every conditional law. Since the set of all stopping times is usually

uncountable, this condition is much stronger, at least formally, than the corresponding property for every fixed stopping time.

We are now ready to give a complete description of the behavior in B_s, and to characterize those σ-fields which yield conditional regeneracy there. In particular, this will enable us to determine precisely when X is regenerative in the entire B.

THEOREM 3. For every stopping time τ, we have

$$P[\theta_\tau \cdot X \in \cdot \mid \mathcal{F}_\tau, X^{-1}\mathcal{J}] = Q^\mu[\cdot \mid \mathcal{J}] \circ X \quad \text{a.s. in } \{X_\tau \in B_s\}, \tag{2.5}$$

and X is a.s. conditionally regenerative in B_s, given $X^{-1}\mathcal{J}$. More generally, a sub-σ-field $\mathcal{C} \subset \mathcal{D}^*$ satisfies

$$P[\theta_\tau \cdot X \in \cdot \mid \mathcal{F}_\tau, X^{-1}\mathcal{C}] = P[\theta_\zeta \circ X \in \cdot \mid X_\zeta, X^{-1}\mathcal{C}] \quad \text{a.s. in } \{X_\tau \in B_s\} \tag{2.6}$$

for all stopping times τ, iff

$$\mathcal{J} \subset \mathcal{C} \vee \sigma(\pi_z) \subset \mathcal{J} \vee \sigma(k_z) \quad \text{a.s. } P\{X \in \cdot, \zeta < \infty\}, \tag{2.7}$$

and in that case the first statement holds with $X^{-1}\mathcal{C}$ in place of $X^{-1}\mathcal{J}$.

COROLLARY. The process X is regenerative in B iff

$$X^{-1}\mathcal{J} = \sigma(X_\zeta) \quad \text{a.s. on } \{\zeta < \infty\}$$

or, equivalently, iff

$$\mathcal{J} = \sigma(\pi_0) \quad \text{a.e. } Q^\mu.$$

Note that the last condition holds automatically when F=S, since in that case Q^μ-a.e. sample path is constant. Thus the last assertion of Theorem 2 follows from the above corollary.

Our final aim is to extend the conditional regeneracy relation (2.5) of Theorem 3 from B_s to the entire B. We shall then employ the notation

$$\mu_\tau = P\{X_\tau \in \cdot, \tau < \infty\},$$

$$Q^{\mu_\tau} = \int Q^x \mu_\tau(dx) = E[Q^{X_\tau}; \tau < \infty],$$

where τ is an arbitrary stopping time.

THEOREM 4. *Let the* σ-*field* $\mathcal{J}' \subset \mathcal{J}$ *be such that* $\mathcal{J}' \vee \sigma(\pi_0) = \mathcal{J}$ *a.e.* Q^{μ}. *Then*

$$P[\theta_\tau \circ X \in \cdot \,|\, \mathcal{F}_\tau,\ X^{-1}\mathcal{J}'] = Q^{\mu_\tau}[\cdot \,|\, \pi_0, \mathcal{J}'] \circ \theta_\tau \circ X \quad \text{a.s. on } \{X_\tau \in B\}, \tag{2.8}$$

for every stopping time τ. *If* $\mathcal{J}'$ *is further countably generated, then* $Q^{\mu_\tau}[\cdot \,|\, \pi_0, \mathcal{J}']$ *can be chosen to be independent of* τ. *The above statements remain true with* $\mathcal{J}'$ *replaced by* $\mathcal{J}'' = \sigma(\theta_z^{-1}\mathcal{J})$.

We do not know whether a.s. conditional regeneracy can be proved here, as in case of Theorem 3. Other difficulties encountered in Theorem 4 are the possible dependence on $\mathcal{J}'$ of the conditional transition kernel in (2.8), and further the possible non-existence of a minimal conditioning σ-field $\mathcal{J}'$. The complexity of the general case is illustrated by Example 1 in Section 8.

We proceed to make some general remarks. First of all, it is easy to check that the proofs below for continuous time carry over with obvious changes to the discrete time case. Alternatively, we may derive the results for discrete time from those in the continuous case by an embedding argument, where we let the successive transitions between states of the discrete time process occur at times given by a homogeneous Poisson process. Note, however, that some technical simplifications are possible when time is discrete. In particular, B_r and B_s will automatically become Borel sets, so there is no need to introduce the auxiliary set B_s'. Furthermore, the statement of Theorem 4 may now be strengtheneed to conditional regeneracy, in the sense discussed above. Apart from this, the two cases are completely analogous. For the sake of simplicity, we shall therefore present the main example of Section 8 in a discrete time version.

For a second remark, it is seen from (2.1) that condition (H) holds a.s. on $\{X_\tau \in B\}$ for a given stopping time τ, iff

$$P\{\theta_\tau \circ X \in \cdot,\ X_\tau \in B\} = E\, Q^{X_\tau}\{w:\ w \in \cdot,\ w_0 \in B\}.$$

Combining this with Theorem 2, we may conclude that the process X is strong Markov, iff

$$P\{\theta_\tau \circ X \in \cdot\} = E\, Q^{X_\tau}$$

for every finite stopping time τ. The latter condition may be easier to verify in specific situations.

For a final remark, note that the local homogeneity of [5] was defined without reference to any specific kernel. In the same vein, one may try to replace H(F,B) by a weaker condition H'(F,B), stating that X is recurrent in F and satisfies

$$P[\theta_\sigma \circ X \in \cdot \,|\, X_\sigma \in dx] = P[\theta_\tau \circ X \in \cdot \,|\, X_\tau \in dx], \qquad x \in B \text{ a.e. } \mu_\sigma \wedge \mu_\tau,$$

for every pair of stopping times σ and τ, such that $X_\sigma, X_\tau \in F$ a.s. (Here $\mu_\sigma \wedge \mu_\tau$ denotes the largest measure dominated by both μ_σ and μ_τ.) In the special case when F=S, we have the following analog of Theorem 2:

THEOREM 2'. <u>Condition</u> H'(S,B) <u>implies that</u> (CI) <u>holds a.s. on</u> $\{X_\tau \in B\}$ <u>for every stopping time</u> τ.

To prove this, one needs to go through the proofs of Theorem 2 and of the first half of Theorem 1, to make sure that the stopping times required to prove the assertions for a fixed τ may be (randomly) selected from some countable family $\{\tau_j\}$. One may then construct a kernel Q such that (H) holds a.s. in B for each τ_j, and then proceed as before. The details of the argument are omitted here.

In the same way, one may prove an analog of Theorem 2 for arbitrary F, though the definition of B_r will now depend on τ, in general. As for the last two main results, a stumbling block seems to be the second half of Theorem 1, whose proof uses stopping times depending in a non-constructive manner on the kernel Q. We do not know how to get around this difficulty.

<u>3. Proof of Theorem 1.</u> Our plan is to prove (2.2), first for stopping times τ with $X_\tau \in F$ a.s. and then in general. We shall next discuss the existence of B_s', and prove from (2.2) that the two sets in (2.3) are indistinguishable. From this the Corollary will follow easily.

PROOF OF (2.2). Fix an arbitrary stopping time τ with $X_\tau \in F$ a.s. Our first aim is to prove that

$$\{X_\tau \in B_s\} \subset \{\theta_\tau \circ X \in A\} \subset \{X_\tau \in B \setminus B_r\} \quad \text{a.s.} \tag{3.1}$$

To see this, conclude from (H) on $\{X_\tau \in B\}$ and from the definitions of B_r and B_s that

$$P\{\theta_\tau \circ X \in A,\ X_\tau \in B_r\} = E\left[Q^{X_\tau}A;\ X_\tau \in B_r\right] = E\left[Q^{X_\tau}A;\ Q^{X_\tau}A=0\right] = 0$$

and

$$P\{\theta_\tau \circ X \notin A,\ X_\tau \in B_s\} = E\left[Q^{X_\tau}A^c;\ X_\tau \in B_s\right] = E\left[Q^{X_\tau}A^c;\ Q^{X_\tau}A=1\right] = 0.$$

The a.s. inclusions in (3.1) follow immediately from these relations and from the definition of A.

Suppose that we can prove the reverse relation

$$\{X_\tau \in B_s\} \supset \{\theta_\tau \circ X \in A\} \quad \text{a.s.} \tag{3.2}$$

Using (H), (2.1) and the definitions of A and B_s, we then obtain

$$\begin{aligned} E\left[Q^{X_\tau}A;\ Q^{X_\tau}A<1\right] &= E\left[Q^{X_\tau}A;\ X_\tau \in B\setminus B_s\right] = P\{\theta_\tau \circ X \in A,\ X_\tau \in B\setminus B_s\} \\ &\le P\{X_\tau \in B_s,\ X_\tau \in B\setminus B_s\} = 0, \end{aligned}$$

which shows that $Q^{X_\tau}A=0$ a.s. on $\{Q^{X_\tau}A<1\}$. By the definitions of B_r and B_s, we get

$$\{X_\tau \in B\setminus B_s\} \subset \{X_\tau \in B_r\} \quad \text{a.s.,}$$

or for the corresponding differences from $\{X_\tau \in B\}$,

$$\{X_\tau \in B_s\} \supset \{X_\tau \in B\setminus B_r\} \quad \text{a.s.}$$

Thus the three events in (3.1) are a.s. equal, and (2.2) follows.

To complete the proof of (2.2) when $X_\tau \in F$ a.s., it remains to prove (3.2). Let us denote the metric in S by ρ, and fix an arbitrary $\varepsilon>0$. We shall first need to show that

$$P\{\theta_\tau \cdot X \notin A,\ X_\tau \in B,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\sigma],\ X_t\in F\} < \delta\} < \varepsilon, \tag{3.3}$$

for sufficiently small constants $\delta>0$ and large stopping times $\sigma\geq\tau$ with $X_\sigma \in F$ a.s. To this aim, note that

$$\sup\{\rho(X_\tau,X_t);\ t\geq\tau,\ X_t\in F\} > 0 \quad \text{on } \{\theta_\tau \circ X\notin A,\ X_\tau\in B\}$$

by the definition of A, and conclude that

$$P\{\theta_\tau \circ X\notin A,\ X_\tau\in B,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\tau\vee s],\ X_t\in F\}<\delta\} < \varepsilon$$

for all sufficiently large $s>0$ and small $\delta>0$. Formula (3.3) follows from this if we replace s by the stopping time $\sigma=\inf\{t\geq\tau\vee s:\ X_t\in F\}$, and it remains to verify that $X_\sigma\in F$ a.s. But this is obvious by the recurrence of F and the facts that F is closed while X is right-continuous.

Returning to the proof of (3.2), we shall introduce a countable collection of auxiliary stopping times τ_j, as follows. First we partition B into countably many disjoint Borel sets B_j with diameters $|B_j|<\delta$, which is possible by the separability of S. We then define, for every index j, an associated stopping time

$$\tau_j = \begin{cases} \inf\{t\in[\tau,\sigma];\ \rho(X_\tau,X_t)\geq\delta,\ X_t\in F\} & \text{when } X_\tau\in B_j, \\ \tau & \text{otherwise.} \end{cases}$$

(Here and below, the infimum of an empty subset of $[\tau,\sigma]$ is taken to be σ.) Note that $X_{\tau_j}\in F$ a.s. Since $|B_j|<\delta$ while X is right-continuous, it is easy to check that

$$\{X_{\tau_j}\in B_j\} \subset \{X_\tau\in B_j,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\sigma],\ X_t\in F\}<\delta\} \tag{3.4}$$

From the definitions of τ_j and A, we further get the relations

$$\{\theta_\tau \cdot X\in A\} \subset \{\theta_{\tau_j}\circ X\in A\} \quad \text{a.s.,} \tag{3.5}$$

$$\{\theta_\tau \circ X\in A,\ X_\tau\in B_j\} \subset \{X_{\tau_j}=X_\sigma=X_\tau\} \quad \text{a.s.} \tag{3.6}$$

Applying (H) to the stopping times τ_j and using formulas (3.3)-(3.6), we obtain

$$E[Q^{X_\tau}A^c;\ \theta_\tau\circ X\in A] = \sum_j E[Q^{X_\tau}A^c;\ \theta_\tau\circ X\in A,\ X_\tau\in B_j]$$

$$\leq \sum_j E[Q^{X_{\tau_j}}A^c;\ X_{\tau_j}\in B_j] = \sum_j P\{\theta_{\tau_j}\circ X\notin A,\ X_{\tau_j}\in B_j\}$$

$$\leq \sum_j P\{\theta_\tau\circ X\notin A,\ X_\tau\in B_j,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\sigma],\ X_t\in F\}<\delta\}$$

$$= P\{\theta_\tau\circ X\notin A,\ X_\tau\in B,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\sigma],\ X_t\in F\}<\delta\} < \varepsilon.$$

Since ε was arbitrary, the left-hand side must be zero, so we get

$$\{\theta_\tau\circ X\in A\} \subset \{Q^{X_\tau}A^c=0\} = \{X_\tau\in B_s\} \qquad \text{a.s.},$$

which is relation (3.2). This completes the proof of (2.2) under the condition $X_\tau\in F$ a.s.

To extend (2.2) to arbitrary stopping times, let us first assume that τ is a.s. finite. Then (2.2) applies to the stopping time $\sigma=\inf\{t\geq\tau:\ X_t\in F\}$ since $X_\sigma\in F$ a.s., and it will then follow for τ if we intersect by $\{X_\tau\in F\}=\{\tau=\sigma\}$. If τ is allowed to be infinite, we may apply (2.4) to the truncated stopping time $\tau\wedge n$ and then intersect by $\{\tau\leq n\}=\{\tau=\tau\wedge n\}$ to obtain

$$\{\theta_\tau\circ X\in A,\ \tau\leq n\} = \{X_\tau\in B_s,\ \tau\leq n\} = \{X_\tau\in B\setminus B_r,\ \tau\leq n\} \qquad \text{a.s.}$$

From this we obtain (2.2) by letting $n\to\infty$. $\square$

We proceed to prove the second statement of Theorem 1. Since this part will not be needed for Theorem 2, the reader may skip the proof for the moment, and return to it in connection with Theorem 3.

As a first step, we shall prove the existence of a Borel set $B_s'\subset B_s$, such that (2.2) remains true with B_s replaced by B_s', i.e. such that

$$\{\theta_\tau\circ X\in A\} = \{X_\tau\in B_s'\} \qquad \text{a.s.} \tag{3.7}$$

for every stopping time τ.

PROOF OF EXISTENCE OF B_s'. Our proof will make use of the measure μ introduced in (2.4). However, that definition uses

the set B_s', so we shall need a direct way of construction. For each $n \in N$ we then introduce the stopping time $\tau_n = \inf\{t \geq n: X_t \in F\}$, and define a measure μ_n on S by

$$\mu_n C = P\{\theta_{\tau_n} \circ X \in A,\ X_{\tau_n} \in C\}, \quad C \in \mathcal{S}. \tag{3.8}$$

The sequence $\{\mu_n\}$ is increasing and bounded, since the events on the right of (3.8) are increasing in n, by the definition of A and the recurrence of F. The measure μ may now be defined by

$$\mu C = \sup_n \mu_n C, \qquad C \in \mathcal{S}.$$

Let us now consider an arbitrary stopping time τ, and conclude as above that, for all $n \in N$ and $C \in \mathcal{S}$,

$$\{\theta_\tau \circ X \in A,\ X_\tau \in C,\ \tau \leq n\} \subset \{\theta_{\tau_n} \circ X \in A,\ X_{\tau_n} \in C\} \quad \text{a.s.}$$

Hence

$$P\{\theta_\tau \circ X \in A,\ X_\tau \in C,\ \tau \leq n\} \leq \mu_n C \leq \mu C.$$

Letting $n \to \infty$ and using (2.2), we get

$$P\{X_\tau \in B_s \cap C\} = P\{\theta_\tau \circ X \in A,\ X_\tau \in C\} \leq \mu C. \tag{3.9}$$

Since $B_s \in \mathcal{S}^*$, there exists some Borel set $B_s' \subset B_s$ satisfying $\mu(B_s \setminus B_s') = 0$. Applying (3.9) with $C = B_s'^{\,c}$, we obtain

$$P\{X_\tau \in B_s \setminus B_s'\} \leq \mu(B_s \setminus B_s') = 0,$$

and (3.7) follows by combination with (2.2). □

To complete the proof of Theorem 1, we have to strengthen (3.7) by proving that the two random sets

$$M_1 = \{t \geq 0:\ \theta_t \circ X \in A\}, \qquad M_2 = \{t \geq 0:\ X_t \in B_s'\}$$

are indistinguishable. If M_1 were known at the outset to be optional, then this would follow directly from the optional section theorem. But the optionality of M_1 is in fact a rather surprising consequence of our theorem.

PROOF OF (2.3). Write D_1 and D_2 for the debuts of the two sets M_1 and M_2. We shall first show that M_1 and M_2 are indistinguishable within the random interval (D_1, ∞). Let us then fix an arbitrary

number $r \geq 0$, and introduce the stopping time $\tau = \inf\{t \geq r: X_t \in F\}$. By the definition of A, we have

$$\tau = \inf\{t \geq r: \theta_t \circ X \in A\} \quad \text{on} \quad \{D_1 < r\}.$$

Since the right-continuity of X implies that M_1 is right-closed, (for a detailed argument, see the proof of the Corollary below), we may conclude that $\theta_\tau \circ X \in A$ on $\{D_1 < r\}$. But then (3.7) yields $X_\tau \in B_s'$ a.s. on $\{D_1 < r\}$, and it follows by the definition of A that M_1 and M_2 are indistinguishable in the random interval $[\tau, \infty) \times \{D_1 < r\}$. Since both sets vanish in $[r, \tau)$ by the definition of τ, they must in fact agree a.s. in $[r, \infty) \times \{D_1 < r\}$. Hence they agree a.s. in the countable union

$$\bigcup_{r \in Q_+} ([r, \infty) \times \{D_1 < r\}) = (D_1, \infty),$$

as asserted.

To extend this result to $[D_1, \infty)$, note that $D_1 \in M_1 \cup \{\infty\}$, since M_1 is right-closed. Using the definition of A, we obtain

$$X_t = X_{D_1}, \qquad t \in M_1 \cap (D_1, \infty),$$

and since M_1 and M_2 agree a.s. on (D_1, ∞), we may conclude that

$$X_{D_1} \in B_s' \quad \text{a.s. on} \ \{M_1 \cap (D_1, \infty) \neq \emptyset\}.$$

But here the set on the right is equivalent to $\{D_1 < \infty\}$, by the recurrence of F, and it follows that $D_1 \in M_2 \cup \{\infty\}$ a.s. Thus M_1 and M_2 are indeed indistinguishable on $[D_1, \infty)$.

It remains to prove that $D_2 > D_1$ a.s. To this aim, let

$$M_\varepsilon = M_2 \cap [D_2, D_2 + \varepsilon), \qquad \varepsilon > 0,$$

and note that M_ε is optional for each ε. The optional section theorem (cf. [4]) then applies and yields the existence, for each $\delta > 0$, of a stopping time τ satisfying

$$[\tau] \subset M_\varepsilon, \qquad P\{\tau = \infty, D_2 < \infty\} < \delta.$$

By (3.7) and the definition of M_ε, we get a.s.

$$\{\tau < \infty\} = \{X_\tau \in B_s', \tau < D_2 + \varepsilon\} = \{\theta_\tau \circ X \in A, \tau < D_2 + \varepsilon\} \subset \{D_1 < D_2 + \varepsilon < \infty\},$$

and hence

$$P\{D_2<\infty\} < P\{\tau<\infty\} + \delta \leq P\{D_1<D_2+\varepsilon<\infty\} + \delta.$$

Since δ was arbitrary, it follows that

$$\{D_2<\infty\} \subset \{D_1<D_2+\varepsilon\} \quad \text{a.s.},$$

and since even ε was arbitrary, we get $D_1 \leq D_2$ a.s. on $\{D_2<\infty\}$. It remains to notice that $D_1 \leq D_2$ is trivially true on $\{D_2=\infty\}$. □

PROOF OF THE COROLLARY. The set M is optional, since X is right-continuous while B_s' is Borel. The last assertion follows immediately from (2.2) and (2.3). It remains to prove that M is a.s. right-closed and either empty or unbounded. By (2.3), it is equivalent to prove these assertions for the set $M'=\{t\geq 0;\ \theta_t \circ X \in A\}$.

Let us then fix an $\omega\in\Omega$, and let $t_1,t_2,\ldots \in M'=M'(\omega)$ with $t_n \downarrow t$. Then $X_{t_1}=X_{t_2}=\ldots=X_t$ by the definition of A and the right-continuity of X, and this is the only state in F which is visited after t. Thus $\theta_t \circ X\in A$, so $t\in M'$. This proves that M' is right-closed.

Let us next fix an $\omega\in\Omega$, such that $M'\neq\emptyset$ while F is recurrent. If $t_0\in M'$, then $X_{t_0}\in B$, and this is the only state in F visited after t_0. Thus X_{t_0} must be visited again for arbitrarily large t, and any such t lies in M' by definition. This proves that M' is unbounded. □

4. Proof of Theorem 2. Here we shall first prove the strong Markov property (SM) in B_r, when F is an arbitrary closed subset of S. As before, it will then be useful to consider the case of stopping times τ with $X_\tau \in F$ a.s., before we turn to the general case. Next we shall assume that F=S, and prove in this case that (SM) remains true in B_s. In view of Theorem 1, this will yield the desired extension to B.

PROOF OF (SM) IN B_r. Assume first that τ is a stopping time with $X_\tau \in F$ a.s. It is required to show that

$$P[\theta_\tau \circ X \in \cdot\,;\ C] = E[Q^{X_\tau};\ C] \tag{4.1}$$

for any set $C \in \mathcal{F}_\tau \cap \{X_\tau \in B_r\}$. For convenience, we shall prove instead that (4.1) holds for any $C \in \mathcal{F}_\tau$ with

$$\{X_\tau \in B \setminus B_r\} \subset C \subset \{X_\tau \in B\}. \tag{4.2}$$

This will prove the original statement, since any set $C \in \mathcal{F}_\tau \cap \{X_\tau \in B_r\}$ may be written as a difference

$$C = (C \cup \{X_\tau \in B \setminus B_r\}) \setminus \{X_\tau \in B \setminus B_r\}$$

between sets in the latter class.

As in the preceding proof, we shall apply condition (H) to a sequence of auxiliary stopping times τ_j. As before, we fix an arbitrary $\varepsilon > 0$, and choose $\delta > 0$ and $\sigma \geq \tau$ such that (3.3) is fulfilled. We shall also need a countable partition of B into disjoint Borel sets B_j with $|B_j| < \delta$. The stopping times required in the present proof are then given by

$$\tau_j = \begin{cases} \inf\{t \in [\tau, \sigma]:\ \rho(X_\tau, X_t) \geq \delta,\ X_t \in F\} & \text{on } C^c \cap \{X_\tau \in B_j\}, \\ \tau & \text{on } C \cup \{X_\tau \notin B_j\}. \end{cases}$$

Recalling the definition of σ and the convention about empty sets, it is seen that $X_{\tau_j} \in F$ a.s., so (H) applies to each τ_j. Note also that, instead of (3.4),

$$\{X_{\tau_j}\in B_j\}\setminus C \subset \{X_\tau\in B,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\sigma],\ X_t\in F\}<\delta\}. \tag{4.3}$$

We may now conclude from (H), (3.3), (4.2) (4.3) and the definition of τ_j that

$$|P[\theta_\tau\circ X\in\cdot;\ C]-E[Q^{X_\tau};\ C]| \le \sum_j |E[1\{\theta_\tau\circ X\in\cdot\}-Q^{X_\tau};\ X_\tau\in B_j,\ C]|$$

$$= \sum_j |E[1\{\theta_{\tau_j}\circ X\in\cdot\}-Q^{X_{\tau_j}};\ X_{\tau_j}\in B_j,\ C]|$$

$$= \sum_j |E[1\{\theta_{\tau_j}\circ X\in\cdot\}-Q^{X_{\tau_j}};\ X_{\tau_j}\in B_j,\ C^c]| \le \sum_j P[X_{\tau_j}\in B_j;\ C^c]$$

$$\le \sum_j P\{\theta_\tau\circ X\notin A,\ X_\tau\in B_j,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\sigma],\ X_t\in F\}<\delta\}$$

$$= P\{\theta_\tau\circ X\notin A,\ X_\tau\in B,\ \sup\{\rho(X_\tau,X_t);\ t\in[\tau,\sigma],\ X_t\in F\}<\delta\}<\varepsilon.$$

Since ε was arbitrary, the difference on the left must vanish, as asserted in (4.1). This proves (SM) in B_r for stopping times τ with $X_\tau\in F$ a.s.

We next consider an arbitrary finite stopping time τ, and a set $C\in\mathcal{F}_\tau\cap\{X_\tau\in B_r\}$. Putting $\sigma=\inf\{t\ge\tau:\ X_t\in F\}$ as before, we get $X_\sigma\in F$ a.s., and it is further seen that

$$C \in \mathcal{F}_\sigma \cap \{\tau=\sigma,\ X_\sigma\in B_r\}. \tag{4.4}$$

Thus (4.1) holds with σ in place of τ, and we get in view of (4.4)

$$P[\theta_\tau\circ X\in\cdot;\ C] = P[\theta_\sigma\circ X\in\cdot;\ C] = E[Q^{X_\sigma};\ C] = E[Q^{X_\tau};\ C],$$

as desired.

For general τ, let again $C\in\mathcal{F}_\tau\cap\{X_\tau\in B_r\}$, and note that

$$C\cap\{\tau\le n\} = C\cap\{\tau=\tau\wedge n\}\in\mathcal{F}_{\tau\wedge n}$$

(cf. [4]), and that moreover

$$C\cap\{\tau\le n\}\subset\{X_\tau\in B_r,\ \tau\le n\}\subset\{X_{\tau\wedge n}\in B_r\}.$$

Hence (4.1) applies with τ and C replaced by $\tau\wedge n$ and $C\cap\{\tau\le n\}$, and we get

$$P[\theta_\tau\circ X\in\cdot,\ \tau\le n;\ C] = P[\theta_{\tau\wedge n}\circ X\in\cdot,\ \tau\le n;\ C] = E[Q^{X_{\tau\wedge n}};\ \tau\le n,\ C]$$

$$= E[Q^{X_\tau};\ \tau\le n,\ C].$$

Since τ is finite on C, we obtain (4.1) from this by letting $n \to \infty$. Our proof is then complete. $\square$

PROOF OF (SM) IN B_s WHEN F=S. If F=S, then

$$A = \{w: w_t = w_0 \in B\} = \{w = c_x : x \in B\},$$

where $c_x(t) \equiv x$, so by Theorem 1 we get for every stopping time τ

$$\theta_\tau \circ X = c_{X_\tau} \quad \text{a.s. on } \{X_\tau \in B_s\}.$$

Hence, a.s. on $\{X_\tau \in B_s\}$,

$$P[\theta_\tau \circ X \in \cdot \mid \mathcal{F}_\tau] = P[c_{X_\tau} \in \cdot \mid \mathcal{F}_\tau] = 1\{c_{X_\tau} \in \cdot\} = P[\theta_\tau \circ X \in \cdot \mid X_\tau],$$

as desired. $\square$

5. Some auxiliary results. In this section we shall collect some auxiliary results, needed for the proofs of the last two main theorems. We shall also introduce the stopping times $\tau_0, \tau_1, \ldots$ and the associated excursions $Y_1, Y_2, \ldots$, which will later play a basic role. The only result in this section which may be of some independent interest is Lemma 5.6 on exchangeable sequences.

Our first aim is to restate the homogeneity condition (H) for visits to B_s' in terms of the mixture Q^{μ}. In the proof, we shall make use of the optional set $M=\{t\geq 0;\ X_t \in B_s'\}$, introduced in Section 2.

LEMMA 5.1. *Let* τ *be a stopping time satisfying* $X_\tau \in B_s'$ *a.s. on* $\{\zeta<\infty\}$. *Then*

$$P\{\theta_\tau \circ X \in \cdot,\ \zeta<\infty\} = Q^{\mu}, \tag{5.1}$$

and

$$P[\theta_\tau \circ X \in \cdot \mid X^{-1}\mathcal{J}] = Q^{\mu}[\cdot \mid \mathcal{J}] \circ X \quad \text{a.s. on} \quad \{\zeta<\infty\}. \tag{5.2}$$

PROOF. To prove (5.1), let us introduce the auxiliary stopping times

$$\tau_n = \inf\{t \geq \tau \wedge n:\ X_t \in F\}, \qquad n \in N,$$

and note that $X_{\tau_n} \in F$ a.s. for each n. Hence (H) applies to each τ_n, and we get

$$P\{\theta_{\tau_n} \circ X \in \cdot,\ X_{\tau_n} \in B_s'\} = E[Q^{X_{\tau_n}};\ X_{\tau_n} \in B_s']. \tag{5.3}$$

Now both $X_\tau \in B_s'$ and $X_{\tau_n} \in B_s'$ imply $\zeta<\infty$, and on the same set $\tau_n = \tau$ for all sufficiently large n. Letting $n\to\infty$ in (5.3), we thus obtain

$$\begin{aligned} P\{\theta_\tau \circ X \in \cdot,\ \zeta<\infty\} &= P\{\theta_\tau \circ X \in \cdot,\ X_\tau \in B_s'\} \\ &= E[Q^{X_\tau};\ X_\tau \in B_s'] = E[Q^{X_\tau};\ \zeta<\infty]. \end{aligned} \tag{5.4}$$

Next conclude from (2.3) that $\theta_\tau \circ X \in A$ a.s. on $\{\zeta<\infty\}$, and that moreover $\zeta = \inf\{t\geq 0:\ \theta_t \circ X \in A\}$ a.s. Since the latter set is right-closed, it follows that even $\theta_\zeta \circ X \in A$ a.s. on $\{\zeta<\infty\}$. Using the definition of A we thus obtain $X_\tau = X_\zeta$ a.s. on $\{\zeta<\infty\}$, and so, by the

definition of μ,

$$E[Q^{X_\tau}; \zeta<\infty] = E[Q^{X_\zeta}; \zeta<\infty] = \int Q^x \mu(dx) = Q^\mu .$$

Combining this with (5.4) yields (5.1).

To prove (5.2), note that $\{\zeta<\infty\}$ a.s. equals $\{M \text{ unbounded}\} \in X^{-1}\mathcal{J}$. For any $I\in\mathcal{J}$ with $\{X\in I\}\subset\{M \text{ unbounded}\}$, we get by (5.1)

$$P\{\theta_\tau\circ X\in\cdot,\ X\in I\} = P\{\theta_\tau\circ X\in\cdot\cap I,\ \zeta<\infty\} = Q^\mu(\cdot\cap I) = \int_I Q^\mu[\cdot|\mathcal{J}]\,dQ^\mu$$

$$= \int_I Q^\mu[\cdot|\mathcal{J}]\,dPX^{-1} = \int_{\{X\in I\}} Q^\mu[\cdot|\mathcal{J}]\circ X\,dP,$$

as desired. □

We proceed to discuss some properties of the random set M. Recall from the Corollary to Theorem 1 that M is a.s. right-closed and either empty or unbounded. We shall also need some information about the gaps of M. By a <u>gap</u> we mean an open interval (a,b), such that $(a,b)\cap M=\emptyset$ while $a,b\in\bar{M}$, where $\bar{M}$ denotes the closure of M. Note that the gaps of M are disjoint.

LEMMA 5.2. <u>For any fixed</u> $h>0$, <u>the random set</u> M <u>has a.s. either none or infinitely many gaps</u> $>h$.

PROOF. Applying Lemma 5.1 to the stopping times

$$\tau_n = \inf\{t\geq n:\ t\in M\},\qquad n\in\mathbb{Z}_+,$$

we get

$$\begin{aligned} P\{M \text{ has gaps} >h\} &= P\{M \text{ has gaps} >h \text{ in } [n,\infty)\} \\ &= P\bigcap_{n=0}\{M \text{ has gaps} >h \text{ in } [n,\infty)\} \\ &= P\{M \text{ has infinitely many gaps} >h\}. \end{aligned}$$

Since the event on the right implies the one on the left, the two must be a.s. equal. Thus the possibility of finitely many gaps $>h$ is a.s. excluded. □

Combining this result with Theorem 1, it is seen that with

probability one either $M=[\zeta,\infty)$ for some $\zeta\in[0,\infty]$, or M has infinitely many gaps greater than some $h>0$. It is convenient to state a corresponding fact in terms of suitable invariant sets in D. Let us then introduce the set-valued function

$$m(w) = \{t\geq 0:\ w_t\in B_s'\},\qquad w\in D,$$

and define the $\mathcal{J}$-sets

$$I_0' = \bigcup_{t\geq 0}\{w\in D:\ m\circ\theta_t w=[t,\infty)\},$$

$$I_h' = \bigcap_{t>0}\{w\in D:\ m\circ\theta_t w \text{ has gaps} > h\},\qquad h>0,$$

$$I_h = I_0'\cup I_h',\qquad h>0;\qquad I_0 = \bigcup_{h>0} I_h.$$

In this notation, we have with probability one either $M=\emptyset$ or $X\in I_0$.

An important role in the sequel will be played by the stopping times $\tau_0,\tau_1,\ldots$ and processes $Y_1,Y_2,\ldots$, defined for fixed $h>0$ by

$$\tau_0=\zeta;\qquad \tau_n = \inf\{t\geq\tau_{n-1}+h:\ t\in M\},\qquad n\in N,$$

$$Y_n(t) = \begin{cases} X(t+\tau_{n-1}), & 0\leq t<\tau_n,\\ \partial, & t\geq\tau_n,\end{cases}\qquad n\in N.$$

Note that the sample paths of $Y_1,Y_2,\ldots$ lie in the space $D_\partial=D(R_+,S_\partial)$, where S_∂ is obtained from S by attaching ∂ as an isolated point.

Let us write Y for the random element $(Y_1,Y_2,\ldots)$ of D_∂^∞. In the latter space, the discrete shift operators θ_n are given by

$$\theta(y_1,y_2,\ldots)=(y_2,y_3,\ldots),\qquad \theta_n=\theta^n.$$

We shall need to construct universally measurable mappings φ and ψ, satisfying $\varphi\circ X=Y$ and $\psi\circ Y=\theta_\zeta\circ X$, and commuting in a suitable sense with the shift operators on D and D_∂^∞.

Let us then define the functions $t_0,t_1,\ldots$ and $\varphi_1,\varphi_2,\ldots$ on D, in the same way as $\tau_0,\tau_1,\ldots$ and $Y_1,Y_2,\ldots$ were defined in terms of X. More precisely, let

$$t_0 = z = \inf m,\qquad d = \inf\{s\geq h;\ s\in m\},$$

$$t_n = t_{n-1} + d\circ\theta_{t_{n-1}},\qquad n\in N,\qquad (5.5)$$

$$\gamma_n = \theta_{t_{n-1}} \circ k_{t_n} = \gamma_1 \circ \theta_{t_{n-1}}, \quad n \in N, \tag{5.6}$$

and put $\gamma = (\gamma_1, \gamma_2, \ldots)$. All these functions are clearly $\mathcal{D}^*$-measurable.

To construct a mapping in the opposite direction, let

$$d'(w) = \inf\{t \geq 0: w_t = \partial\}, \quad w \in D_\partial,$$

and define for $y = (y_1, y_2, \ldots) \in D_\partial^\infty$

$$r_n(y) = d'(y_1) + \ldots + d'(y_n), \quad n \in Z_+,$$

$$(\psi(y))_t = \begin{cases} y_n(t - r_{n-1}), & r_{n-1} \leq t < r_n, \quad n \in N, \\ \partial, & t \geq \sup r_n. \end{cases}$$

LEMMA 5.3. <u>The above functions are universally measurable and satisfy</u>

$$\gamma \circ \theta_{t_n} = \theta_n \circ \gamma, \quad n \in Z_+, \tag{5.7}$$

$$\psi \circ \theta_n \circ \gamma = (\theta_{r_n} \circ \psi) \circ \gamma = \theta_{t_n}, \quad n \in Z_+. \tag{5.8}$$

<u>Moreover, there exist universally measurable functions</u> $n_t, n_t': I_h' \to Z_+$, $t \geq 0$, <u>such that</u>

$$\theta_{n_t} \circ \gamma = \theta_{n_t'} \circ \gamma \circ \theta_t \quad \underline{\text{on}}\ I_h, \quad t \geq 0. \tag{5.9}$$

PROOF. For every $t \geq 0$ we have

$$t + d \circ \theta_t = t + \inf\{s \geq h: s \in m \circ \theta_t\} = \inf\{s + t \geq h + t: s + t \in m\}$$
$$= \inf\{s \geq h + t: s \in m\},$$

so

$$z \circ \theta_{t + d \circ \theta_t} = 0, \quad t \geq 0.$$

Putting $t = t_{n-1}$, we get in particular

$$z \circ \theta_{t_n} = 0 \text{ on the set } \{t_n < \infty\}.$$

More generally, it is seen by induction that

$$t_k \circ \theta_{t_n} + t_n = t_{k+n}, \quad k, n \in Z_+. \tag{5.10}$$

Indeed, assuming (5.10) to be true with k-1 instead of k, we get by (5.5)

$$t_k \circ \theta_{t_n} + t_n = (t_{k-1} + d \circ \theta_{t_{k-1}}) \circ \theta_{t_n} + t_n$$
$$= t_{k-1} \circ \theta_{t_n} + t_n + d \circ \theta_{t_{k-1} \circ \theta_{t_n} + t_n}$$
$$= t_{k-1+n} + d \circ \theta_{t_{k-1+n}} = t_{k+n},$$

as desired. Combining (5.10) with (5.6) yields

$$\varphi_k \circ \theta_{t_n} = \varphi_1 \circ \theta_{t_{k-1} \circ \theta_{t_n}} \circ \theta_{t_n} = \varphi_1 \circ \theta_{t_{k-1} \circ \theta_{t_n} + t_n} = \varphi_1 \circ \theta_{t_{k+n-1}} = \varphi_{k+n},$$

and (5.7) follows.

To prove (5.8), note that

$$t_{n-1} + d' \circ \varphi_n = t_n, \quad n \in N,$$

and conclude by summation that

$$r_n \circ \varphi = t_n - z \quad \text{on } \{z < \infty\}, \quad n \in Z_+. \tag{5.11}$$

Assuming that $z < \infty$ and $t+z \in [t_{n-1}, t_n)$, we get

$$(\psi \circ \varphi)_t = \varphi_n(t - r_{n-1} \circ \varphi) = \varphi_n(t + z - t_{n-1}) = w(t+z) = \theta_z(t),$$

so

$$\psi \circ \varphi = \theta_z, \tag{5.12}$$

at least when $z < \infty$. But then (5.12) must be generally true, since it reduces to a triviality when $z = \infty$. Combining (5.12) with (5.7) and (5.10) yields

$$\psi \circ \theta_n \circ \varphi = \psi \circ \varphi \circ \theta_{t_n} = \theta_{z \circ \theta_{t_n}} \circ \theta_{t_n} = \theta_{z \circ \theta_{t_n} + t_n} = \theta_{t_n},$$

which proves the equality between the extreme members of (5.8). On the other hand, we may use (5.11) and (5.12) to obtain for $z < \infty$

$$(\theta_{r_n} \circ \psi) \circ \varphi = \theta_{r_n \circ \varphi} \circ \psi \circ \varphi = \theta_{t_n - z} \circ \theta_z = \theta_{t_n},$$

which again extends immediately to arbitrary z. Thus the right equality in (5.8) is also true, and so the proof of (5.8) is complete.

To prove (5.9) for suitable n_t and n_t', let us first assume that $w \in I_h'$, and define

$$T(w) = \inf\{s \geq 0 : [s, s+h] \cap m(w) \neq \emptyset\},$$

$$n_t(w) = \inf\{n\in Z_+ : t_n > T\circ\theta_t + t\}, \quad t\geq 0.$$

Note that T and n_t are finite and $\mathfrak{D}^*$-measurable functions of w on I_h'. We shall show that

$$t_{n_t+k} = t + t_{n_0\circ\theta_t+k}\circ\theta_t, \quad k\in Z_+, \ t\geq 0. \tag{5.13}$$

To see this, conclude from the definitions that

$$t_{n_t} = \inf\{s\in m : s > T\circ\theta_t + t\}, \quad t\geq 0.$$

Hence

$$\begin{aligned} t_{n_0\circ\theta_t}\circ\theta_t &= \inf\{s\in m\circ\theta_t,\ s > T\circ\theta_t\} \\ &= \inf\{s\geq 0 : s+t\in m,\ s+t > T\circ\theta_t + t\} \\ &= \inf\{u\in m : u > T\circ\theta_t + t\} - t = t_{n_t} - t, \end{aligned}$$

which proves (5.13) for k=0. Combining this special case with (5.10), we obtain more generally

$$\begin{aligned} t_{n_t+k} = t_{n_t} + t_k\circ\theta_{t_{n_t}} &= t + t_{n_0\circ\theta_t}\circ\theta_t + t_k\circ\theta_{t+t_{n_0\circ\theta_t}} \\ &= t + t_{n_0\circ\theta_t}\circ\theta_t + t_k\circ\theta_{t_{n_0\circ\theta_t}}\circ\theta_t = t + t_{n_0\circ\theta_t+k}\circ\theta_t, \end{aligned}$$

as desired. Using (5.13), we get for $k\in N$

$$\begin{aligned} \varphi_{k+n_0\circ\theta_t}\circ\theta_t &= \theta_t((\cdot+t_{k+n_0\circ\theta_t-1})\wedge t_{k+n_0\circ\theta_t}) \\ &= w((\cdot+t_{k+n_t-1})\wedge t_{k+n_t}) = \varphi_{k+n_t}, \end{aligned}$$

so (5.9) holds with

$$n_t'(w) = n_0(\theta_t w), \quad t\geq 0.$$

It remains to consider the case when $w\in I_0'$. Let us then define

$$T'(w) = \inf\{s\geq 0 : w_s\in B,\ \theta_s w = c_{w_s}\},$$

$$N(w) = \inf\{k\in Z_+ : t_k(w) > T'(w)\},$$

and note that N is finite and $\mathfrak{D}^*$-measurable. Moreover, $w_s = w_\infty$ for $s\geq t_N$ while $(\theta_t w)_s = w_\infty$ for $s\geq t_{N\circ\theta_t}\circ\theta_t$, so we get

$$\varphi_{N+k} = \varphi_{N\circ\theta_t+k}\circ\theta_t, \quad k\in N,\ t\geq 0.$$

Thus (5.9) holds in this case with

$$n_t(w) = N(w), \quad n_t'(w) = N(\theta_t w), \quad t \geq 0. \qquad \square$$

It is convenient to restate (5.7) and (5.8) in terms of X and Y:

COROLLARY 5.4. <u>For each</u> $n \in Z_+$, <u>we have identically</u>

$$\varphi \cdot \theta_{\tau_n} \circ X = \theta_n \circ Y, \tag{5.14}$$

$$\psi \cdot \theta_n \circ Y = \theta_{\tau_n} \circ X. \tag{5.15}$$

Our next aim is to relate the shift invariant σ-fields in D and D_∂^∞ through the mapping φ. In D_∂^∞, the invariant σ-field is given by

$$\mathcal{J} = \{J \in \mathcal{D}^{*\infty}: \theta^{-1}J = J\}.$$

To stress the dependence of $\varphi, Y, \ldots$ on h, we shall sometimes write $\varphi_h, Y_h, \ldots$

LEMMA 5.5. <u>For each</u> $h > 0$ <u>we have</u>

$$\mathcal{I} \cap I_0 \subset \varphi_h^{-1} \mathcal{J}. \tag{5.16}$$

<u>Moreover, there exists some set</u> $J_h \in \mathcal{J}$, <u>such that</u>

$$\mathcal{I} \cap I_h = \varphi_h^{-1}(\mathcal{J} \cap J_h). \tag{5.17}$$

PROOF. Let $I \in \mathcal{I} \cap I_0$ be arbitrary, and put

$$J = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \theta_n^{-1} \psi^{-1} I.$$

Then clearly $J \in \mathcal{J}$. Using Lemma 5.3, the invariance of I and the finiteness of $t_1, t_2, \ldots$ on I_0, it is further seen that

$$\{\varphi \in \theta_n^{-1} \psi^{-1} I\} = \{\psi \circ \theta_n \circ \varphi \in I\} = \{\theta_{t_n} \in I\} = I,$$

so

$$\{\varphi \in J\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\varphi \in \theta_n^{-1} \psi^{-1} I\} = I.$$

This proves (5.16).

From (5.16) it follows in particular that $I_h = \varphi^{-1} J_h$ for some $J_h \in \mathcal{J}$. To prove (5.17), it remains to show that

$$\varphi^{-1}(\mathcal{J} \cap J_h) \subset \mathcal{I} \cap I_h.$$

Let us then consider an arbitrary set $J \in \mathcal{J} \cap J_h$, and put $I = \varphi^{-1} J$. Then

$I \subset I_h$, so we need only prove that $I \in \mathcal{J}$. Using Lemma 5.3 and the invariance of J, we get for any $t \geq 0$ and for n_t and n_t' as in (5.9)

$$\theta_t^{-1} I = \theta_t^{-1}\varphi^{-1} J = \{\varphi \circ \theta_t \in J\} = \{\theta_{n_t'} \circ \varphi \circ \theta_t \in J\} = \{\theta_{n_t} \circ \varphi \in J\} = \{\varphi \in J\}$$
$$= \varphi^{-1} J = I,$$

as desired. □

We conclude this section with a general result about conditionally i.i.d. random sequences in an arbitrary state space S. Let us then write θ for the shift operator on S^∞ and $\mathcal{J}$ for the associated σ-field of shift invariant measurable sets in S^∞.

LEMMA 5.6. <u>Let</u> $\xi = (\xi_0, \xi_1, \ldots)$ <u>be a random sequence such that</u> $\theta \circ \xi$ <u>is conditionally i.i.d. and independent of</u> ξ_0, <u>given some</u> σ-<u>field</u> $\mathcal{G}$. <u>Then this remains true with</u> $\mathcal{G}$ <u>replaced by</u> $\xi^{-1}\mathcal{C}$, <u>iff</u>

$$\theta^{-1}\mathcal{J} \subset \mathcal{C} \subset \theta^{-1}\mathcal{J} \vee \sigma(\pi_0) \quad \underline{\text{a.s.}}\ P\xi^{-1}. \tag{5.18}$$

Note that this statement contains a number of known results in exchangeability theory (cf. Aldous [1]).

PROOF. By hypothesis,

$$P[\theta \circ \xi \in \cdot \,|\, \xi_0, \mathcal{G}] = (P[\xi_1 \in \cdot \,|\, \mathcal{G}])^\infty = \nu^\infty. \tag{5.19}$$

Hence, by the law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1\{\xi_j \in B\} = \nu B \quad \text{a.s.},$$

so $\sigma(\nu) \subset \xi^{-1}\theta^{-1}\mathcal{J}$ a.s., and therefore

$$P[\theta \circ \xi \in \cdot \,|\, \xi_0,\ \xi^{-1}\theta^{-1}\mathcal{J}] = (P[\xi_1 \in \cdot \,|\, \xi_0,\ \xi^{-1}\theta^{-1}\mathcal{J}])^\infty$$
$$= (P[\xi_1 \in \cdot \,|\, \xi^{-1}\theta^{-1}\mathcal{J}])^\infty.$$

Thus (5.19) remains true for $\mathcal{G} = \xi^{-1}\mathcal{C}$, with any σ-field $\mathcal{C}$ satisfying (5.18).

Suppose conversely that (5.19) holds with $\mathcal{G} = \xi^{-1}\mathcal{C}$ for some σ-field $\mathcal{C}$. By Kolmogorov's 0-1 law, we obtain for any $I \in \mathcal{J}$

$$P[\theta\xi\in I\,|\,\xi^{-1}\mathcal{C}] = (P[\xi_1\in\cdot\,|\,\xi^{-1}\mathcal{C}])^{\infty}I = 0 \text{ or } 1 \text{ a.s.},$$

and it follows easily that, a.s.,

$$\{\theta\xi\in I\} = \{P[\theta\xi\in I\,|\,\xi^{-1}\mathcal{C}]=1\} \in \xi^{-1}\mathcal{C}.$$

Thus $\xi^{-1}\theta^{-1}\mathcal{J}\subset\xi^{-1}\mathcal{C}$ a.s., so $\theta^{-1}\mathcal{J}\subset\mathcal{C}$ a.s. $P\xi^{-1}$.

To derive the upper bound in (5.18), note that (5.19) remains true with $\mathcal{G}=\sigma(\xi_0)\vee\xi^{-1}\mathcal{C}$. Applying the law of large numbers to successive blocks of n components, we get a.s.

$$P[(\xi_1,\dots,\xi_n)\in\cdot\,|\,\xi_0,\ \xi^{-1}\mathcal{C}] = P[(\xi_1,\dots,\xi_n)\in\cdot\,|\,\xi_0,\ \xi^{-1}\theta^{-1}\mathcal{J}],$$

and hence also

$$P[\xi_0\in\cdot,\ (\xi_1,\dots,\xi_n)\in\cdot\,|\,\xi_0,\xi^{-1}\mathcal{C}]=P[\xi_0\in\cdot,\ (\xi_1,\dots,\xi_n)\in\cdot\,|\,\xi_0,\xi^{-1}\theta^{-1}\mathcal{J}].$$

By a monotone class argument, this extends to

$$P[\xi\in\cdot\,|\,\xi_0,\xi^{-1}\mathcal{C}] = P[\xi\in\cdot\,|\,\xi_0,\xi^{-1}\theta^{-1}\mathcal{J}],$$

and we get in particular, for any $C\in\mathcal{C}$,

$$1\{\xi\in C\} = P[\xi\in C\,|\,\xi_0,\xi^{-1}\mathcal{C}] = P[\xi\in C\,|\,\xi_0,\xi^{-1}\theta^{-1}\mathcal{J}] \in \sigma(\xi_0)\vee\xi^{-1}\theta^{-1}\mathcal{J}.$$

Thus $\xi^{-1}\mathcal{C}\subset\sigma(\xi_0)\vee\xi^{-1}\theta^{-1}\mathcal{J}$ a.s., so $\mathcal{C}\subset\sigma(\pi_0)\vee\theta^{-1}\mathcal{J}$ a.s. $P\xi^{-1}$. □

6. Proof of Theorem 3. The main idea of the proof is to show by means of Theorem 2.1 in [5] that the sequence of excursions $Y_1, Y_2, \ldots$ is exchangeable and hence conditionally i.i.d. This yields the conditional form of the strong Markov property at the random times $\tau_0, \tau_1, \ldots$ Since an arbitrary stopping time in M can be approximated by times τ_n, the general result follows from this by a continuity argument. Similar arguments were used on several occations already in [5].

We proceed to the detailed argument, and begin with (2.5). Here it will be convenient to consider a special case first.

PROOF OF (2.5) WHEN $X \in I_h$ A.S. Let $\mathcal{G}$ be the discrete filtration associated with the stopping times $\tau_0, \tau_1, \ldots$ above, i.e. $\mathcal{G}_n = \mathcal{F}_{\tau_n}$ for $n \in Z_+$, and note that Y is adapted to $\mathcal{G}$. If σ is a finite $\mathcal{G}$ stopping time, then τ_σ is an $\mathcal{F}$ stopping time, and moreover $X_{\tau_\sigma} \in B_s'$ a.s. since M is right-closed. Hence

$$P\{\theta_{\tau_\sigma} \in \cdot\} = Q^\mu$$

by Lemma 5.1, so by Corollary 5.4 we have

$$P\{\theta_\sigma Y \in \cdot\} = P\{\varphi \circ \theta_{\tau_\sigma} X \in \cdot\} = Q^\mu \varphi^{-1},$$

independently of σ. We may then conclude from Theorem 2.1 in [5] that Y is exchangeable, and that indeed

$$P[\theta_n Y \in \cdot \mid \mathcal{G}_n, \nu] = \nu^\infty = P[Y \in \cdot \mid \nu] \quad \text{a.s.}$$

for some random probability measure ν on D_∂. This extends by a standard argument to

$$P[\theta_\sigma Y \in \cdot \mid \mathcal{G}_\sigma, \nu] = P[Y \in \cdot \mid \nu] \quad \text{a.s.,}$$

for finite $\mathcal{G}$ stopping times σ. Noting that

$$\sigma(\nu) = Y^{-1}\mathcal{J} = X^{-1}\mathcal{J} \quad \text{a.s.}$$

by Lemmas 5.5 and 5.6, and using Lemma 5.1 and Corollary 5.4, we hence obtain

$$P[\theta_{\tau_\sigma}\circ X\in\cdot\,|\,\mathcal{F}_{\tau_\sigma},\ X^{-1}\mathcal{J}] = P[\psi\circ\theta_\sigma\circ Y\in\cdot\,|\,\mathcal{H}_\sigma, \text{[illegible]}] = P[\psi\circ Y\in\cdot\,|\,\text{[illegible]}]$$

$$= P[\theta_\zeta\circ X\in\cdot\,|\,X^{-1}\mathcal{J}] = Q^\mu[\cdot\,|\,\mathcal{J}]\circ X.$$

This proves (2.5) for stopping times τ of the form τ_σ.

To extend this to arbitrary stopping times τ with $X_\tau\in B_s'$ a.s., put

$$\tau_h = \inf\{\tau_n\geq\tau:\ n\in Z_+\},$$

and note that τ_h is a.s. of the form τ_σ for some $\mathcal{G}$ stopping time σ. Thus (2.5) holds with τ_h in place of τ, and since $\mathcal{F}_\tau\subset\mathcal{F}_{\tau_h}$, we get

$$P[\theta_{\tau_h}\circ X\in\cdot\,|\,\mathcal{F}_\tau,\ X^{-1}\mathcal{J}] = Q^\mu[\cdot\,|\,\mathcal{J}]\circ X \quad \text{a.s.}$$

This clearly remains true for every sufficiently small $h>0$.

Now $\tau_h\to\tau$ a.s. as $h\to 0$ by construction, so we get

$$P[X(\tau_h+t)\ \to\ X(\tau+t)\,|\,\mathcal{F}_\tau,\ X^{-1}\mathcal{J}] = 1 \quad \text{a.s.,}\quad t\geq 0,$$

by the right-continuity of paths. Hence the finite-dimensional distributions of $\theta_{\tau_h}\circ X$ converge weakly to those of $\theta_\tau\circ X$, a.s. with respect to $P[\cdot\,|\,\mathcal{F}_\tau,\ X^{-1}\mathcal{J}]$. On the other hand, these distributions are a.s. given by $Q^\mu[\cdot\,|\,\mathcal{J}]\circ X$. Thus both sides of (2.5) have the same finite-dimensional distributions, and since the latter determine the whole distribution, (2.5) is generally true.

Let us next consider the case when τ is an arbitrary finite stopping time. We then define $\sigma=\inf\{t\geq\tau:\ X_t\in B_s'\}$, and note that σ is a stopping time with $X_\sigma\in B_s'$ a.s., since M is a.s. right-closed. Hence (2.5) holds with τ replaced by σ, and since $\mathcal{F}_\sigma\cap\{\sigma=\tau\}=\mathcal{F}_\tau\cap\{\sigma=\tau\}$, the original relation must be a.s. true on $\{\sigma=\tau\}$. It remains to notice that $\{X_\tau\in B_s'\}\subset\{\sigma=\tau\}$ by the definition of σ.

We finally assume that τ is an arbitrary stopping time. Then (2.5) holds with τ replaced by $\tau\wedge n$ for any fixed $n\in N$, and since $\mathcal{F}_{\tau\wedge n}\cap\{\tau\leq n\}=\mathcal{F}_\tau\cap\{\tau\leq n\}$, the original relation is then a.s. true on $\{X_\tau\in B_s',\ \tau\leq n\}$. The assertion for τ now follows by letting $n\to\infty$. $\square$

PROOF OF (2.5) IN THE GENERAL CASE. From the assertions in Lemma 5.1, we get immediately the corresponding statements for the conditional probabilities $P[\cdot\,|X\in I_h]$ and $Q^\mu[\cdot\,|I_h]=Q^\mu(\cdot\cap I_h)/Q^\mu I_h$, provided that $P\{X\in I_h\}>0$. Proceeding as in the special case above, it is then seen that

$$P[\theta_\tau\cdot X\in\cdot\,|\mathcal{F}_\tau,\ X^{-1}\mathcal{J},\ X\in I_h]=Q^\mu[\cdot\,|\mathcal{J},I_h]\circ X \text{ a.s. on } \{X_\tau\in B_s',\ X\in I_h\},$$

which is equivalent to (2.5) on the set $\{X\in I_h\}$. Since h is arbitrary, this extends immediately to $\{X\in I_0\}$. It remains to notice that $\{X_\tau\in B_s'\}\subset\{X\in I_0\}$ a.s., by Lemma 5.2. □

PROOF OF THE CONDITIONAL REGENERACY IN B_s. We shall prove the stronger statement that, for almost every conditional distribution $P'=P[\cdot\,|X^{-1}\mathcal{J}]$, there exists some probability measure Q' on D, such that

$$P'[\theta_\tau\circ X\in\cdot\,|\mathcal{F}_\tau]=Q' \quad\text{on } \{X_\tau\in B_s'\},\qquad \text{a.s. } P', \tag{6.1}$$

for every stopping time τ. By (2.5), this holds with $Q'=Q^\mu[\cdot\,|\mathcal{J}]\circ X$ for every fixed stopping time τ. The point is that the exceptional P-null set where (6.1) may fail can be taken to be independent of τ.

To see this, note that (6.1) holds simultaneously for the countable collection of stopping times $\tau_{0,k}=\zeta$ and

$$\tau_{n,k}=\inf\{t\geq\tau_{n-1,k}+k^{-1},\ t\in M\},\qquad n,k\in\mathbb{N},$$

outside a fixed P-null set. For any P' and Q' with this property, we may now proceed as above to extend (6.1) in steps, first to stopping times of the form $\tau_{\sigma,k}$ with σ random, then to stopping times τ with $X_\tau\in B_s'$ a.s. P', next to all finite stopping times, and finally to the general case. □

From (2.5) follows the same relation with $X^{-1}\mathcal{C}$ in place of $X^{-1}\mathcal{J}$, for arbitrary $\mathcal{C}$ satisfying (2.7), and (2.6) then follows by applying (2.5) to the stopping times τ and ζ. To complete the

proof of Theorem 3, it thus remains to show that (2.6) for all stopping times τ implies (2.7).

PROOF OF (2.7) FROM (2.6). Fix $h>0$, define $Y=(Y_1,Y_2,\ldots)$ as before, and put $Y_0=k_\zeta\circ X$. From (2.6) it is seen that $Y_1,Y_2,\ldots$ are conditionally i.i.d. and independent of Y_0, a.s. on $\{\zeta<\infty\}$ and given the σ-field $X^{-1}\mathcal{C}\vee\sigma(X_\zeta)$. By Lemma 5.6, this implies

$$Y^{-1}\mathcal{J}\subset X^{-1}\mathcal{C}\vee\sigma(X_\zeta)\subset Y^{-1}\mathcal{J}\vee\sigma(Y_0)\qquad\text{a.s. on }\{\zeta<\infty\}. \tag{6.2}$$

In particular we may take $\mathcal{C}=\mathcal{J}$, and then we get in conjunction with Lemma 5.5 and Theorem 1

$$X^{-1}\mathcal{J}\subset Y^{-1}\mathcal{J}\subset X^{-1}\mathcal{J}\vee\sigma(X_\zeta)=X^{-1}\mathcal{J}\qquad\text{a.s. on }\{\zeta<\infty\}.$$

Thus $Y^{-1}\mathcal{J}=X^{-1}\mathcal{J}$ a.s. on $\{\zeta<\infty\}$, and (2.7) follows by substituting $X^{-1}\mathcal{J}$ for $Y^{-1}\mathcal{J}$ in (6.2). □

PROOF OF THE COROLLARY. From Theorems 2 and 3 it is seen that X is regenerative in B iff

$$\mathcal{J}\subset\sigma(\pi_z)\subset\mathcal{J}\vee\sigma(k_z)\quad\text{a.s. }P\{X\in\cdot,\ \zeta<\infty\}. \tag{6.3}$$

Now π_z is clearly shift invariant on the set where $\theta_z\in A$ while m is unbounded, and the latter set has full measure, since

$$P\{\theta_\zeta\circ X\in A,\ M\text{ is unbounded}\}=P\{\zeta<\infty\},$$

by Theorem 1 and its corollary. Thus

$$\sigma(\pi_z)\subset\mathcal{J}\quad\text{a.s. }P\{X\in\cdot,\ \zeta<\infty\},$$

so (6.3) reduces to

$$\mathcal{J}=\sigma(\pi_z)\ \text{a.s. }P\{X\in\cdot,\ \zeta<\infty\}.$$

Using the definition of $\mathcal{J}$, we may rewrite this in the form

$$\mathcal{J}=\sigma(\pi_0)\quad\text{a.s. }P\{\theta_\zeta\circ X\in\cdot,\ \zeta<\infty\}=Q^\mu.$$ □

7. Proof of Theorem 4. Our plan is first to derive (2.8) from the three preceding theorems. Our next step is to construct a τ-independent version of $Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']$ from a product measurable version of $Q^{\cdot}[\cdot|\mathcal{J}']$, where the latter is known to exist when $\mathcal{J}'$ is countably generated. To prove the final assertion, we shall show that $\mathcal{J}''$ is equivalent to some σ-field satisfying the previous requirements, in the sense of yielding the same conditional distributions in (2.8).

PROOF OF (2.8). Our first aim is to prove that (2.8) holds a.s. on $\{X_\tau\in B_r\}$ for any stopping time τ. To see this, fix a stopping time τ, let $I\in\mathcal{J}'$ and $C\in\mathcal{S}^*\cap B_r$ be arbitrary, and conclude from the invariance of I and the definitions of Q^{μ_τ} and $Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']$ that

$$
\begin{aligned}
P\{\theta_\tau\circ X\in\cdot,\ X\in I,\ X_\tau\in C\} &= P\{\theta_\tau\circ X\in\cdot\cap I\cap\pi_0^{-1}C\} = Q^{\mu_\tau}(\cdot\cap I\cap\pi_0^{-1}C)\\
&= \int Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']\,1(I\cap\pi_0^{-1}C)\,dQ^{\mu_\tau}\\
&= E[Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']\circ\theta_\tau\circ X;\ X\in I,\ X_\tau\in C].
\end{aligned}
$$

Hence

$$P[\theta_\tau\circ X\in\cdot,\ X\in I|X_\tau] = E[Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']\circ\theta_\tau\circ X;\ X\in I|X_\tau]\ \text{a.s. on}\ \{X_\tau\in B_r\}.$$

Since (SM) holds in B_r by Theorem 2, it follows that

$$P[\theta_\tau\circ X\in\cdot,\ X\in I|\mathcal{F}_\tau] = E[Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']\circ\theta_\tau\circ X;\ X\in I|\mathcal{F}_\tau]\ \text{a.s. on}\ \{X_\tau\in B_r\}.$$

Thus we get, a.s. on $\{X_\tau\in B_r\}$,

$$
\begin{aligned}
P[\theta_\tau\circ X\in\cdot|\mathcal{F}_\tau,\ X^{-1}\mathcal{J}'] &= E\, Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']\circ\theta_\tau\circ X|\mathcal{F}_\tau,\ X^{-1}\mathcal{J}']\\
&= Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']\circ\theta_\tau\circ X,
\end{aligned}
$$

as asserted.

By Theorem 1 it remains to prove (2.8) a.s. on $\{X_\tau\in B_s\}$, and by Theorem 3 it then suffices to show that

$$Q^{\mu_\tau}[\cdot|\pi_0,\mathcal{J}']\circ\theta_\tau\circ X = Q^{\mu}[\cdot|\mathcal{J}]\circ X\quad \text{a.s. on}\ \{X_\tau\in B_s\}. \tag{7.1}$$

To see this, note that $X_\tau=X_\zeta$ a.s. on $\{X_\tau\in B_s\}$. Hence $\mu_\tau\le\mu$ on B_s,

and therefore $Q^{\mu_\tau} \leq Q^{\mu}$ on $\pi_0^{-1} B_s$. Thus we get, by the hypothesis on $\mathcal{J}'$

$$Q^{\mu_\tau}[\cdot \mid \pi_0, \mathcal{J}'] = Q^{\mu}[\cdot \mid \pi_0, \mathcal{J}'] = Q^{\mu}[\cdot \mid \mathcal{J}] \quad \text{a.e. } Q^{\mu_\tau} \text{ on } \pi_0^{-1} B_s,$$

and (7.1) follows by the definitions of $\mathcal{J}$ and Q^{μ_τ}. □

For the next step of our proof, we shall need the following standard result (cf. [4]):

LEMMA 7.1. <u>Let</u> P^s <u>be a probability kernel from a measurable space</u> $(S, \mathcal{S})$ <u>into some Polish space</u> $(T, \mathcal{T})$, <u>and let</u> $\mathcal{T}'$ <u>be a countably generated sub-</u>σ<u>-field of</u> $\mathcal{T}$. <u>Then</u> $P^s[\cdot \mid \mathcal{T}'](t)$ <u>has a regular and</u> $\mathcal{S} \times \mathcal{T}'$<u>-measurable version.</u>

We may now prove the second statement of the theorem.

PROOF OF THE τ-INDEPENDENCE. If $\mathcal{J}'$ is countably generated, then Lemma 7.1 yields the existence of a regular and product measurable version of the conditional probabilities $Q^{\cdot}[\cdot \mid \mathcal{J}']$. Given any such version, we shall prove that

$$Q^{\mu_\tau}[\cdot \mid \pi_0, \mathcal{J}'](w) = Q^{w_0}[\cdot \mid \mathcal{J}'](w) \quad \text{a.e. } Q^{\mu_\tau}, \tag{7.2}$$

for every stopping time τ. Since the right-hand side is $\sigma(\pi_0) \vee \mathcal{J}'$-measurable, a τ-independent version of $Q^{\mu_\tau}[\cdot \mid \mathcal{J}']$ may then be <u>defined</u> by the equality in (7.2).

To prove (7.2), note first that $E(\xi f) = (E\xi) f$ for any random measure ξ and measurable function $f \geq 0$ on the same space. Using this fact, along with the definitions of Q^{μ_τ} and $Q^{\cdot}[\cdot \mid \mathcal{J}']$ and the normality of $Q^{\cdot}$, we get for arbitrary $I \in \mathcal{J}'$ and $C \in \mathcal{S}$

$$\begin{aligned} Q^{\mu_\tau}(\cdot \cap I \cap \pi_0^{-1} C) &= E\, Q^{X_\tau}(\cdot \cap I \cap \pi_0^{-1} C) = E[Q^{X_\tau}(\cdot \cap I);\ X_\tau \in C] \\ &= E[\int_I Q^{X_\tau}[\cdot \mid \mathcal{J}']\, dQ^{X_\tau};\ X_\tau \in C] \\ &= E \int_{I \cap \pi_0^{-1} C} Q^{w_0}[\cdot \mid \mathcal{J}'](w)\, Q^{X_\tau}(dw) \end{aligned}$$

$$= \int_{I \cap \tau_0^{-1} C} Q^{w_0}[\ \mathcal{J}'](w)\ Q^{\mu_\tau}(dw),$$

as desired. □

To prove the last statement of the theorem, let us introduce the space $M(D_\partial)$ of probability measures on D_∂, and let $\mathcal{M}$ be the σ-field in $M(D_\partial)$ generated by the mappings $m \to mC$, $m \in M(D_\partial)$, for arbitrary $C \in \mathcal{D}_\partial = \mathcal{D} \vee \{\partial\}$. We shall need another standard fact:

LEMMA 7.2. The σ-field $\mathcal{M}$ in $M(D_\partial)$ is countably generated.

PROOF. Note that $\mathcal{D}_\partial$ is the Borel σ-field in D_∂ generated by the Skorohod-Stone topology. Similarly, $\mathcal{M}$ is the Borel σ-field in $M(D_\partial)$ generated by the topology of weak convergence for probability measures on D_∂. Since S_∂ is Polish, so is D_∂ and hence also $M(D_\partial)$. In particular, $\mathcal{M}$ is then generated by any countable base in $M(D_\partial)$. □

PROOF OF THE LAST STATEMENT. Fix positive numbers $h_n \downarrow 0$, and write

$$J_1 = I_{h_1}; \qquad J_n = I_{h_n} \setminus I_{h_{n-1}}, \qquad n=2,3,\ldots,$$

where the sets I_h, $h \geq 0$, are defined as in Section 5. Let ν_n be the random probability measure ν of Section 6 corresponding to $h=h_n$, and conclude from the law of large numbers that ν_n has a $Y_{h_n}^{-1}\mathcal{J}$-measurable version. Since moreover $Y_{h_n}^{-1}\mathcal{J} = X^{-1}\mathcal{J}$ on $\{X \in J_n\}$ by Lemma 5.5, there exists an $\mathcal{J}/\mathcal{M}$-measurable mapping $m_n : J_n \to M(D)$, such that $\nu_n = m_n \circ X$ a.s. on $\{X \in J_n\}$. Let $\mathcal{J}'$ be the σ-field in D generated by the mappings $m_1, m_2, \ldots$

By construction we have $\mathcal{J}' \subset \mathcal{J}''$, and from Lemma 7.2 it follows that $\mathcal{J}'$ is countably generated. By Lemma 5.6 it is further seen that $X^{-1}\mathcal{J}' = X^{-1}\mathcal{J}$ on each set $\{X \in J_n\}$, so we get

$$(\theta_\zeta \circ X)^{-1}\mathcal{J}' = X^{-1}\mathcal{J}' = X^{-1}\mathcal{J} = (\theta_\zeta \circ X)^{-1}\mathcal{J} \quad \text{a.s. on } \{\zeta < \infty\}, \tag{7.3}$$

and hence $\mathcal{J}'=\mathcal{J}$ a.e. Q^{μ}. Thus $\mathcal{J}'$ fulfills the hypotheses of the theorem, so (2.8) must hold with some τ-independent version of $Q^{\mu\tau}[\cdot\,|\,\pi_0,\mathcal{J}']$.

To complete the proof, it suffices to show that

$$P[\theta_\tau \circ X \in \cdot\,|\,\mathcal{F}_\tau,\ X^{-1}\mathcal{J}'] = P[\theta_\tau \circ X \in \cdot\,|\,\mathcal{F}_\tau,\ X^{-1}\mathcal{J}''] \quad \text{a.s.}$$

and

$$Q^{\mu\tau}[\cdot\,|\,\pi_0,\mathcal{J}'] = Q^{\mu\tau}[\cdot\,|\,\pi_0,\mathcal{J}''] \quad \text{a.e. } Q^{\mu\tau} \text{ on } \pi_0^{-1}S,$$

in the sense that any version on the right is also a version on the left. Since $\mathcal{J}''\subset\mathcal{J}'$, it is then enough to show that

$$X^{-1}\mathcal{J}' = X^{-1}\mathcal{J}'' \quad \text{a.s.} \tag{7.4}$$

and

$$\mathcal{J}' = \mathcal{J}'' \quad \text{a.e. } Q^{\mu\tau} \text{ on } \pi_0^{-1}S. \tag{7.5}$$

Here (7.4) follows from (7.3) and from the fact that $\{\zeta=\infty\}$ is a.s. an atom of both σ-fields. Given (7.4), we get as before

$$(\theta_\tau \circ X)^{-1}\mathcal{J}' = X^{-1}\mathcal{J}' = X^{-1}\mathcal{J}'' = (\theta_\tau \circ X)^{-1}\mathcal{J}'' \quad \text{a.s. on } \{\tau<\infty\},$$

which implies (7.5). $\square$

8. Examples. The main purpose of the present section is to illustrate the results of Section 2 by a simple example, exhibiting most of the features of the general case. Here we are choosing the time scale to be discrete, for convenience, but it is easy to construct an analogous example in continuous time, by letting the transitions between states occur at times given by a homogeneous Poisson process.

Example 1. Let $S=\{0,1,...,8\}$, and let Ω consist of the following six "paths" from Z_+ to S:

$$\omega_1=0171717..., \quad \omega_3=0383838..., \quad \omega_5=0555...,$$
$$\omega_2=0272727..., \quad \omega_4=0484848..., \quad \omega_6=0666...$$

Let $\mathcal{F}$ be the discrete σ-field on Ω, and let $\mathcal{F}_0, \mathcal{F}_1, ...$ be the filtration generated by the identity map $X=(X_0, X_1, ...)$ on Ω. Note that $\mathcal{F}_0=\{\emptyset, \Omega\}$ while $\mathcal{F}_1=\mathcal{F}_2=...=\mathcal{F}$. On Ω we introduce the probability measure P, assigning the same probability 1/6 to all paths.

Using the fact that stopping times with respect to $\{\mathcal{F}_n\}$ are either identically zero or strictly positive, one can easily show that $H(F,B)$ is true with $F=\{0,5,6,7,8\}$ and $B=\{0,7,8\}$. Here $Q^0=P$, while Q^7 assigns probability 1/2 to each one of $\theta_2\omega_1$ and $\theta_2\omega_2$, and similarly for Q^8. Thus $B_r=\{0\}$ while $B_s=\{7,8\}$, and it is easily seen that X regenerates in B_r but not in B_s. This illustrates the statements of Theorems 1 and 2.

Turning to the statements of Theorem 3, note that Q^μ gives mass 1/4 to each of the paths $\theta_2\omega_1, ..., \theta_2\omega_4$. Since these are separated by invariant sets, it follows that $Q^\mu[\cdot\,|\mathcal{J}](\omega_j)$ degenerates at $\theta_2\omega_j$ for $j=1,...,4$. Since moreover $X^{-1}\mathcal{J}=\mathcal{F}$ in the present case, it is seen that $P[\cdot\,|X^{-1}\mathcal{J}](\omega_j)$ degenerates at ω_j for $j=1,...,4$. Thus (2.5) holds as asserted. However, the two equivalent conditions

of the Corollary fail in the present case, since $\sigma(X_\zeta)\cap\{\zeta<\infty\}$ is the σ-field generated by the partition $(\{\omega_1,\omega_2\},\ \{\omega_3,\omega_4\})$, while $X^{-1}\mathcal{J}\cap\{\zeta<\infty\}$ is the one generated by $(\{\omega_1\},\ldots,\{\omega_4\})$.

Most interesting in the present example is perhaps to look at the statements of Theorem 4. Since S is finite, we can always choose

$$Q^{\mu\tau}[\cdot\,|\,\pi_0,\mathcal{J}'](w) = Q^{w_0}[\cdot\,|\,\mathcal{J}'](w), \qquad w=\omega_1,\ldots,\omega_6,\ \theta_2\omega_1,\ldots,\theta_2\omega_4,$$

regardless of the nature of $\mathcal{J}'$. In particular, (2.8) holds with $Q^{\mu\tau}$ as above for $\mathcal{J}'=\mathcal{J}$, but also for $\mathcal{J}'$ equal to, say,

$$\mathcal{J}_1' = \sigma(I_{13},I_{24}) \quad \text{or} \quad \mathcal{J}_2' = \sigma(I_{14},I_{23}),$$

where I_{ij} denotes the set of paths visiting the set $\{i,j\}$ infinitely often. For the latter, $Q^7[\cdot\,|\,\mathcal{J}']$ and $Q^8[\cdot\,|\,\mathcal{J}']$ are a.s. degenerate as before, but $Q^0[\cdot\,|\,\mathcal{J}_j']$ is non-degenerate for j=1,2, and differs for the two cases. In fact, it is easily verified that $Q^0[\omega_i\,|\,\mathcal{J}_1'](\omega_j) =1/2$ for $i,j\in\{1,3\}$ or $i,j\in\{2,4\}$, and similarly for $\mathcal{J}_2'$ with 3 and 4 interchanged. Note also that both $\mathcal{J}_1'$ and $\mathcal{J}_2'$ are locally minimal, in the sense that, whenever a σ-field $\mathcal{J}'\subset\mathcal{J}_j'$ is such that $\mathcal{J}'\vee\sigma(\pi_0)=\mathcal{J}$ a.e. Q^μ, then $\mathcal{J}'=\mathcal{J}_j'$ a.e. Q^μ. This illustrates the non-uniqueness of kernel and non-existence of a minimal conditioning σ-field in the statement of Theorem 4. □

We conclude this section by correcting an error in [5] related to the present work. As part of Theorem 4.5 in [5], it was claimed that a real valued, continuous, recurrent, and locally homogeneous process X on R_+ is conditionally strong Markov. However, the proof in [5] is false in general, unless we exclude the possibility of paths with constant pieces. (Our mistake was to add up uncountably many null-sets, corresponding to the possible states of constancy.) The following counterexample shows that the difficulties are intrinsic rather than merely technical.

Example 2. Consider a Brownian motion B, and an independent homogeneous Poisson process with jumps at $\tau_1 < \tau_2 < \ldots$ Put $\tau_0=0$, and define

$$X_t = \sum_{k=0}^{\infty} B(((t-k) \vee \tau_k) \wedge \tau_{k+1}) - B(\tau_k) \quad , \qquad t \geq 0.$$

Thus X consists of diffusion parts of independent exponential lengths, alternating with constant parts of length 1.

Since the levels of constancy $B(\tau_1), B(\tau_2), \ldots$ have diffuse distributions, it is easily seen that X regenerates at every fixed state. But X is not Markov or conditionally Markov, even in the weak sense, since every fixed time $t > 0$ belongs with positive probability to some interval of constancy (τ_k+k-1, τ_k+k). $\square$

REFERENCES

[1] ALDOUS, D.J. (1985). Exchangeability and related topics. Ecole d'été de probabilités de Saint-Flour XIII -1983, pp. 1-198. Lecture Notes in Mathematics 1117. Springer, Berlin.

[2] BLUMENTHAL, R.M. and GETOOR, R.K. (1968). Markov Processes and Potential Theory. Academic Press, New York.

[3] DELLACHERIE, C. (1978). Quelques examples familiers, en probabilités, d'ensembles analytiques, non boréliens. Séminaire de Probabilités XII, Lecture Notes in Mathematics 649 746-756, Springer, Berlin.

[4] DELLACHERIE, C. and MEYER, P.A. (1975, 1980). Probabilités et Potentiel, Chap. I-VIII. Hermann, Paris.

[5] KALLENBERG, O. (1982a). Characterizations and embedding properties in exchangeability. Z. Wahrscheinlichkeitstheorie verw. Gebiete 60 249-281.

[6] KALLENBERG, O. (1982b). A dynamical approach to exchangeability. Exchangeability in Probability and Statistics (ed. by G. Koch and F. Spizzichino), pp. 87-96. North-Holland, Amsterdam.

[7] WALSH, J.B. (1972). Transition functions of Markov processes. Séminaire de Probabilités VI. Lecture Notes in Mathematics 258 215-232. Springer, Berlin.

DTIC
END
4-86